Exactly solvable models through the empty-interval method

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(Received 6 May 2001; published 24 October 2001)

The most general one dimensional reaction-diffusion model with nearest-neighbor interactions that can be solved exactly through empty-interval method has been introduced. Assuming translationally invariant initial conditions, the probability that n consecutive sites are empty, E_n, has been exactly obtained. Here, however, we do not consider reactions changing two empty neighboring sites. In the thermodynamic limit, the large-time behavior of the system has also been investigated. Releasing translationally invariance, the evolution equation for the probability that n consecutive sites, starting from the site k, are empty, $E_{k,n}$, is obtained. In the thermodynamic limit, the large time behavior of the system is also considered. Finally, the continuum limit of the model is considered and the empty-interval probability function is obtained.

DOI: 10.1103/PhysRevE.64.056116 PACS number(s): 02.50.Ga, 82.40.Bj, 05.40.−a

I. INTRODUCTION

The principles of equilibrium statistical mechanics are well established. But, thermal equilibrium is a special case and little is known about the properties of systems not in equilibrium, for example, about the relaxation toward the stationary state. There is no general approach to systems far from equilibrium. As mean-field techniques, generally, do not give correct results for low-dimensional systems, people are motivated to study exactly solvable stochastic models in low dimensions. Moreover, solving one-dimensional systems should, in principle, be easier. Different methods have been used to study these models, including analytical and asymptotic methods, mean-field methods, and large-scale numerical methods. Exact results for some models on a one-dimensional lattice have been obtained, for example, in [1–16]. The term exactly solvable has been used with different meanings.

In Ref. [17] a ten-parameter family of reaction-diffusion processes was introduced for which the evolution equation of n-point functions contains only n or less point functions. The average particle number in each site has been obtained exactly for these models. In Ref. [19] the same method has been used to analyze the above mentioned ten-parameter family model on finite lattice with boundaries. As another approach, the term integrablility in Ref. [18] means that the N-particle conditional probabilities' S matrix is factorized into a product of two-particle S matrices, and for multispecies models, the two-particle S matrix satisfies the quantum Yang-Baxter equation.

The empty-interval method (EIM) has been used to analyze the one-dimensional dynamics of diffusion-limited coalescence [20–23]. Using this method, the probability that n consecutive sites are empty has been calculated. This method has been used to study a reaction-diffusion process with three-site interactions [24]. EIM has been also generalized to study the kinetics of the q-state one-dimensional Potts model in the zero-temperature limit [25].

In this paper, we are going to study all the one-dimensional reaction-diffusion models with nearest-neighbor interactions that can be exactly solved by EIM. It is worth noting that ben-Avraham and coworkers have studied one-dimensional diffusion-limited processes through EIM [20–23]. In their study, some of the reaction rates have been taken infinite, and they have worked out the models on continuum. Some results for the discrete lattice have been obtained in [25]. For the cases of finite reaction rates, some approximate solutions have been obtained.

We study models with finite reaction rates, obtain conditions for the system to be solvable via EIM, and then solve the equations of EIM. We do this for a system on a lattice and on continuum.

The scheme of the paper is as follows. In Sec. II the most general one-dimensional reaction-diffusion model with nearest-neighbor interactions that can be solved exactly through EIM has been introduced. Assuming translational invariance, the probability that n consecutive sites are empty, E_n, has been exactly obtained. Here, however, we do not consider reactions changing two empty neighboring sites. In the thermodynamic limit, the large-time behavior of the system has also been investigated. In Sec. III the assumption of translational invariance has been released and the evolution equation for the probability that n consecutive sites, starting from the site k, are empty, $E_{k,n}$, is obtained. In the thermodynamic limit, the large-time behavior of the system is also considered. It is shown that translationally asymmetric fluctuations relative to the stationary configuration disappear faster than the translationally symmetric fluctuations. In Sec. IV the continuum limit of the model is considered and the empty-interval probability function is obtained.

II. MODELS SOLVABLE THROUGH THE EMPTY-INTERVAL METHOD: THE TRANSLATIONALLY INVARIANT CASE

Consider a general one-species reaction-diffusion model on a one-dimensional lattice with $L+1$ sites, with nearest-

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neighbor interactions. We want to impose restrictions on the reaction and diffusion rates so that the system is solvable via EIM, that is, so that the evolution equation for the probability that n consecutive sites are empty E_n is closed. Suppose that the initial condition of the system is translationally invariant. The most general interactions for a single-species model in a one-dimensional lattice with nearest-neighbor interactions are

$$\bullet\circ\rightarrow(\bullet\bullet,\ \circ\bullet,\ \circ\circ)\quad \circ\bullet\rightarrow(\bullet\circ,\ \bullet\bullet,\ \circ\circ) \tag{1}$$

$$\bullet\bullet\rightarrow(\bullet\circ,\ \circ\circ,\ \circ\bullet)\quad \circ\circ\rightarrow(\bullet\circ,\ \bullet\bullet,\ \circ\bullet),$$

where an empty (occupied) site is denoted by $\circ$ ($\bullet$). The constraint of solvability of the model through EIM, imposes that, as we will show, there are no processes in which the final configuration is $\circ\circ$. We shall also see that the processes the initial configuration of them is $\circ\circ$, have no effect on the solvability through EIM. But first, let us consider only the systems for them there are no interactions with $\circ\circ$ as the initial or final configuration. So, among the above 12 interactions, only the following six interactions remain to be considered:

$$\bullet\circ\rightarrow\begin{cases}\bullet\bullet, & r_1\\ \circ\bullet, & r_2\end{cases},\quad \circ\bullet\rightarrow\begin{cases}\bullet\circ, & r_3\\ \bullet\bullet, & r_4\end{cases},$$

$$\bullet\bullet\rightarrow\begin{cases}\bullet\circ, & r_5\\ \circ\bullet, & r_6\end{cases}. \tag{2}$$

The parameters r_i are the rate of interactions. Define

$$P(\overbrace{\circ\circ\cdots\circ}^{n})=E_n, \tag{3}$$

from which, one obtains

$$P(\bullet\overbrace{\circ\circ\cdots\circ}^{n})=3DP(\overbrace{\circ\circ\cdots\circ}^{n}\bullet)=E_n-E_{n+1}, \tag{4}$$

where $\bullet$ ($\circ$) indicates an occupied (empty) site, and P denotes the probability of the configuration. The evolution equation for $E_n(t)$ is

$$\begin{aligned}\frac{dE_n}{dt}=&r_5P(\bullet\overbrace{\bullet\circ\cdots\circ}^{n})+r_3P(\circ\overbrace{\bullet\circ\cdots\circ}^{n})\\&+r_6P(\overbrace{\circ\cdots\circ\bullet}^{n}\bullet)+r_2P(\overbrace{\circ\cdots\circ\bullet}^{n}\circ)-(r_1+r_2)\\&\times P(\bullet\overbrace{\circ\circ\cdots\circ}^{n})-(r_3+r_4)P(\overbrace{\circ\cdots\circ\circ}^{n}\bullet).\end{aligned} \tag{5}$$

The right-hand side of Eq. (5) is not generally in terms of only E_n's. In order to make it expressible in terms of E_n's, using Eq. (4), one has to impose

$$r_3=r_5,\quad r_2=r_6. \tag{6}$$

Then, using

$$\begin{aligned}P(\bullet\overbrace{\bullet\circ\cdots\circ}^{n})+P(\circ\overbrace{\bullet\circ\cdots\circ}^{n})&=P(\overbrace{\bullet\circ\cdots\circ}^{n})\\&=E_{n-1}-E_n,\end{aligned} \tag{7}$$

and another similar relation, one arrives at

$$\begin{aligned}\frac{dE_n(t)}{dt}=&(r_2+r_3)(E_{n-1}+E_{n+1}-2E_n)\\&-(r_1+r_4)(E_n-E_{n+1}),\quad n>1.\end{aligned} \tag{8}$$

Note that if there were reactions with the final configuration $\circ\circ$ then one encounters with terms like $P(\circ\circ\bullet\circ\cdots\circ)$ or $P(\circ\circ\bullet\bullet\circ\cdots\circ)$ at the right-hand side of Eq. (5), which are not expressible in terms of E_k's. On the other hand, if $\circ\circ$ is the initial configuration in a reaction, the evolution equation for E_n's is still closed, although the analog of Eq. (8) will be a linear finite difference equation with nonconstant coefficients. We do not consider this reaction here.

The equation of motion of $E_1(t)$ is

$$\frac{dE_1(t)}{dt}=(r_2+r_3)(1+E_2-2E_1)-(r_1+r_4)(E_1-E_2). \tag{9}$$

It is seen that it takes a form similar to Eq. (8), provided one defines

$$E_0(t):=1. \tag{10}$$

Then we have Eq. (8), for $n\geqslant 1$, equipped with the boundary condition (10). We also set $E_{L+1}(t)=0$, which means that initially at least one particle is present in the lattice. If initially all the sites were empty [$E_n(0)=1$ for all n], then the above-defined reactions would not change the configuration of the system, and if initially at least one particle was present, then at any time $E_{L+1}(t)=0$. So, the completely empty lattice is a stationary state that is decoupled from any other state. Defining

$$b:=\frac{r_1+r_4}{r_2+r_3} \tag{11}$$

and rescaling the time properly, the equation of motion becomes

$$\begin{aligned}\frac{dE_n(t)}{dt}=&E_{n-1}+E_{n+1}-2E_n+b(E_{n+1}-E_n),\\&0<n<L+1\end{aligned} \tag{12}$$

with the boundary conditions

$$E_0(t)=1,\quad E_{L+1}(t)=0. \tag{13}$$

A particular solution to this is the stationary solution

$$E^{\mathrm{P}}_{n-1}+E^{\mathrm{P}}_{n+1}-2E^{\mathrm{P}}_{n}+b(E^{\mathrm{P}}_{n+1}-E^{\mathrm{P}}_{n})=0. \tag{14}$$

Taking the ansatz

$$E^{\mathrm{P}}_{n}=Az_1^n+Bz_2^n, \tag{15}$$

for E^{P}_n, one arrives at

$$z_i+z_i^{-1}-2+b(z_i-1)=0, \tag{16}$$

the solutions of which are $z_1=1/(1+b)$ and $z_2=1$. Using the boundary conditions $E_0=1$ and $E_{L+1}=0$, A and B are obtained as

$$A=\frac{1}{1-(1+b)^{-L-1}}, \qquad B=\frac{-(1+b)^{-L-1}}{1-(1+b)^{-L-1}}. \tag{17}$$

Defining

$$F_n(t):=E_n(t)-E^{\mathrm{P}}_n, \tag{18}$$

it is seen that the evolution equation for F_n is the same as that of E_n, but the boundary conditions for F_n are homogeneous. The initial condition for F_n is

$$F_n(0)=E_n(0)-E^{\mathrm{P}}_n. \tag{19}$$

To calculate $F_n(t)$, one seeks the eigenvalues and eigenvectors of the operator at the right-hand side of Eq. (12), that is

$$\epsilon f_n=f_{n+1}+f_{n-1}-2f_n+b[f_{n+1}-f_n]. \tag{20}$$

The solution to this is

$$f_n=az_1^n+b'z_2^n, \tag{21}$$

where z_i's satisfy

$$z_i^2(1+b)-z_i(\epsilon+2+b)+1=0. \tag{22}$$

Now, defining

$$Z_i:=z_i\sqrt{1+b}, \tag{23}$$

it is seen $Z_1Z_2=1$. So

$$f_n=\frac{1}{(1+b)^{n/2}}(aZ^n+b'Z^{-n}). \tag{24}$$

The boundary conditions $F_0(t)=F_{L+1}(t)=0$, lead to

$$Z=\exp\left(\frac{i\pi k}{L+1}\right), \tag{25}$$

where k is an integer satisfying $1<k<L+1$, and

$$f_{k,n}=\frac{1}{(1+b)^{n/2}}\sin\left(\frac{\pi nk}{L+1}\right). \tag{26}$$

The corresponding eigenvalue is then

$$\epsilon_k=-2-b+2\sqrt{1+b}\cos\left(\frac{\pi k}{L+1}\right). \tag{27}$$

Then $F_n(t)$ will be

$$F_n(t)=\sum_{k=1}^{L}\frac{\alpha_k}{(1+b)^{n/2}}\sin\left(\frac{\pi nk}{L+1}\right)e^{\epsilon_k t}, \tag{28}$$

where

$$\alpha_k=\frac{2}{L+1}\sum_{m=1}^{L}[E_m(0)-E^{\mathrm{P}}_m](1+b)^{m/2}\sin\left(\frac{m\pi k}{L+1}\right). \tag{29}$$

In the thermodynamic limit ($L\to\infty$), $F_n(t)$ takes a simpler form. Defining $x:=\pi k/(L+1)$, Eqs. (28) and (29) lead to

$$\begin{aligned}F_n(t)&=\frac{2}{\pi}\sum_{m=1}^{\infty}\int_0^{\pi}dx\,\exp[(-2-b+2\sqrt{1+b}\cos x)t]\\&\quad\times\sin(nx)\sin(mx)F_m(0)(1+b)^{(m-n)/2}\\&=\sum_{m=1}^{\infty}(1+b)^{(m-n)/2}e^{-(2+b)t}[\mathrm{I}_{m-n}(2t\sqrt{1+b})\\&\quad-\mathrm{I}_{m+n}(2t\sqrt{1+b})][E_m(0)-E^{\mathrm{P}}_m],\end{aligned} \tag{30}$$

where in the second line we have used the integral representation of the modified Bessel functions

$$\mathrm{I}_n(t)=\frac{1}{\pi}\int_0^{\pi}dx\cos(nx)e^{t\cos x}. \tag{31}$$

To study the large-time behavior of $F_n(t)$, one takes $\alpha_k=:\pi B_k/(L+1)$. In the thermodynamic limit ($L\to\infty$), Eq. (28) leads to

$$\begin{aligned}F_n(t)&=\frac{1}{(1+b)^{n/2}}\int_0^{\pi}dxB(x)\\&\quad\times\exp[(-2-b+2\sqrt{1+b}\cos x)t]\sin(nx).\end{aligned} \tag{32}$$

At large times, the main contribution to the integral comes from the region $x\approx 0$, in which the exponent of the exponential term takes its largest value. So,

$$\begin{aligned}F_n(t)&\approx\frac{n}{(1+b)^{n/2}}\exp[(-2-b+2\sqrt{1+b})t]\\&\quad\times\int_0^{\pi}dx\;\;e^{-\sqrt{1+b}x^2t}xB(x),\end{aligned} \tag{33}$$

or

$$F_n(t) \sim \frac{\exp[(-2-b+2\sqrt{1+b})t]}{t}, \tag{34}$$

provided $B(x)$ is well behaved and nonzero at $x=0$. If $b \neq 0$, there is an energy gap in the spectrum and the system relaxes towards its stationary state exponentially. If $b=0$, there is no energy gap and the relaxation towards the stationary state is in the form of power law with the exponent -1. The dependence of F_n on b shows that increasing the value of b leads to a sharper decrease of F_n with respect to n. This can be seen from Eq. (33). This is reasonable, since increasing b means increasing the creation rate or decreasing the annihilation rate. This prevents formation of large empty intervals.

The empty-interval probability functions can be used to obtain some kinds of n-point functions. It is easy to see that

$$P(\bullet\overbrace{\circ\circ\cdots\circ}^{m}\bullet) = E_m - 2E_{m+1} + E_{m+2}. \tag{35}$$

So, the results for $E_n(t)$'s can be used to obtain the probability that between two occupied sites, there are n sites, which are empty. The one-point function and the two-point function $\langle n_i n_{i+1}\rangle$ can also be obtained using E_n's. In fact, $\langle n\rangle = 1 - E_1$, and $\langle n_i n_{i+1}\rangle = E_2 - 2E_1 + 1$.

III. MODELS SOLVABLE THROUGH THE EMPTY-INTERVAL METHOD: THE GENERAL CASE

In the preceding section, we considered translationally invariant initial conditions. As the dynamics is translationally invariant, the probability $E_n(t)$ will be the same for all sites, provided the initial condition for it is so. In this section, we release the translational invariance of the initial conditions and the quantity of our interest is the probability $E_{k,n}(t)$, that n consecutive sites, starting from the site k are empty at time t,

$$P_k(\overbrace{\circ\circ\cdots\circ}^{n}) = E_{k,n} \tag{36}$$

It is easy to see that

$$\begin{aligned} P_l(\bullet\overbrace{\circ\circ\cdots\circ}^{m}) &= E_{l+1,m} - E_{l,m+1}, \\ P_l(\overbrace{\circ\circ\cdots\circ}^{m}\bullet) &= E_{l,m} - E_{l,m+1}. \end{aligned} \tag{37}$$

Using the interactions (2) and the above identities, one arrives at

$$\begin{aligned} \frac{dE_{k,n}(t)}{dt} = {} & r_3(E_{k+1,n-1} - E_{k,n}) + r_2(E_{k,n-1} - E_{k,n}) \\ & - (r_1 + r_2)(E_{k,n} - E_{k-1,n+1}) \\ & - (r_3 + r_4)(E_{k,n} - E_{k,n+1}). \end{aligned} \tag{38}$$

Similar to the preceding section, imposing the boundary condition

$$E_{k,0}(t) = 1 \tag{39}$$

makes the evolution equation valid for $0 < n < L+1$. If the lattice is not initially empty, one also has

$$E_{k,L+1}(t) = 0. \tag{40}$$

Using the definitions

$$\begin{aligned} b &:= \frac{r_1 + r_4}{r_2 + r_3}, \\ c &:= \frac{r_1 + r_2}{r_2 + r_3}, \\ d &:= \frac{r_3}{r_2 + r_3}, \end{aligned} \tag{41}$$

Eq. (38) can be rearranged in the form

$$\begin{aligned} \frac{dE_{k,n}(t)}{dt} = {} & E_{k,n-1} + E_{k,n+1} - 2E_{k,n} - b(E_{k,n} - E_{k,n+1}) \\ & - c(E_{k,n+1} - E_{k-1,n+1}) + d(E_{k+1,n-1} - E_{k,n-1}). \end{aligned} \tag{42}$$

Using the particular solution E_n^P, one defines

$$F_{k,n}(t) := E_{k,n}(t) - E_n^P, \tag{43}$$

which satisfies Eq. (42), but with homogeneous boundary conditions,

$$F_{k,0}(t) = F_{k,L+1}(t) = 0. \tag{44}$$

Applying the Fourier transformation

$$\tilde{F}_n(\omega,t) := \sum_k \omega^k F_{k,n}(t), \tag{45}$$

one arrives at

$$\begin{aligned} \frac{d\tilde{F}_n(\omega,t)}{dt} = {} & \tilde{F}_{n-1}(\omega,t) + \tilde{F}_{n+1}(\omega,t) - 2\tilde{F}_n(\omega,t) \\ & - b[\tilde{F}_n(\omega,t) - \tilde{F}_{n+1}(\omega,t)] - c[\tilde{F}_{n+1}(\omega,t) \\ & - \omega\tilde{F}_{n+1}(\omega,t)] + d[\omega^{-1} = \tilde{F}_{n-1}(\omega,t) \\ & - \tilde{F}_{n-1}(\omega,t)]. \end{aligned} \tag{46}$$

To solve this, one first solves the eigenvalue problem

$$\begin{aligned} \epsilon\tilde{f}_n(\omega) = {} & \tilde{f}_{n-1}(\omega) + \tilde{f}_{n+1}(\omega) - 2\tilde{f}_n(\omega) \\ & - b[\tilde{f}_n(\omega) - \tilde{f}_{n+1}(\omega)] \\ & - c[\tilde{f}_{n+1}(\omega) - \omega\tilde{f}_{n+1}(\omega)] \\ & + d[\omega^{-1}\tilde{f}_{n-1}(\omega) - \tilde{f}_{n-1}(\omega)]. \end{aligned} \tag{47}$$

using the ansatz

$$\tilde{f}_n = a z_1^n + b' z_2^n, \tag{48}$$

and the boundary conditions, one arrives at

$$\tilde{f}_{s,n}(\omega) = [B(\omega)]^{n/2} \sin\left(\frac{n\pi s}{L+1}\right)$$

$$\epsilon_s = -2 - b + D(\omega)\cos\left(\frac{\pi s}{L+1}\right), \tag{49}$$

where

$$D(\omega) := 2\sqrt{[1+d(\omega^{-1}-1)][1+b+c(\omega-1)]},$$

$$B(\omega) := \frac{1+d(\omega^{-1}-1)}{1+b+c(\omega-1)}, \tag{50}$$

and s is an integer between 1 and L. In the thermodynamic limit $L \to \infty$, one arrives at

$$\tilde{F}_n(\omega,t) = \sum_m \tilde{F}_m(\omega,0) \times [B(\omega)]^{(n-m)/2} e^{-(2+b)t} \{ \mathrm{I}_{m-n}[D(\omega)t] - \mathrm{I}_{m+n}[D(\omega)t] \}, \tag{51}$$

where

$$\tilde{F}_m(\omega,0) = \sum_k \omega^k [E_{k,m}(0) - E_m^P]. \tag{52}$$

Now, let us consider the relaxation of the system towards its stationary state. Suppose that the initial value for $E_{k,n}$ is so that $\tilde{F}_n(\omega,0)$ contains a term proportional to $\delta(p)$ (where $\omega = e^{ip}$), and another term that is a smooth function of ω. The δ term comes from a translationally invariant part in $F_{n,k}(0)$. Using the steepest descent method, one can see that the relaxation behavior of the second term is governed by the extremum value of the eigenvalues ϵ with respect to a complex ω. This is found to be

$$\epsilon_{\max} = -2 - b + 2[\sqrt{dc} + \sqrt{(1-d)(1+b-c)}]. \tag{53}$$

It is easy to show

$$\epsilon_{\max} \leqslant -2 - b + 2\sqrt{1+b}. \tag{54}$$

Equality holds when

$$r_2(r_1 + r_2) = r_3(r_3 + r_4). \tag{55}$$

This means that the relaxation time for the translationally noninvariant part is smaller than of the translationally invariant part. That is, the translationally noninvariant fluctuations disappear faster than the translationally invariant parts.

IV. THE CONTINUUM LIMIT

In the preceding section, we considered the probability of finding n consecutive empty sites starting from the kth site. For the continuum limit, it is better to use a quantity with arguments symmetric relative to the starting point and the end point of the empty interval. For an empty interval of the length n starting from the site k, the end site is $k' = k + n - 1$. Then one can use

$$s := k + k' = 2k + n - 1 \tag{56}$$

instead of k for labeling the empty interval. $s/2$ is the center of the empty interval. So, one uses the quantity

$$\mathcal{E}_{s,n}(t) := E_{k,n}(t). \tag{57}$$

The equation of motion for $\mathcal{F}_{s,n}(t)$ is then

$$\frac{d\mathcal{F}_{s,n}(t)}{dt} = r_2(\mathcal{F}_{s-1,n-1} + \mathcal{F}_{s-1,n+1} - 2\mathcal{F}_{s,n}) + r_3(\mathcal{F}_{s+1,n-1} + \mathcal{F}_{s+1,n+1} - 2\mathcal{F}_{s,n}) + r_4(\mathcal{F}_{s+1,n+1} - \mathcal{F}_{s,n}) + r_1(\mathcal{F}_{s-1,n+1} - \mathcal{F}_{s,n}). \tag{58}$$

Here $\mathcal{F}$ is the solution to the evolution equation of $\mathcal{E}$, but with homogeneous boundary conditions. Using $X := s/2$ and $x := n$, and Taylor expanding the above expression in the continuum limit, one arrives at

$$\frac{\partial \mathcal{F}(X,x;t)}{\partial t} = \left(A\partial_X + B\partial_x + \frac{C}{4}\partial_X^2 + C\partial_x^2 + D\partial_x\partial_X \right) \times \mathcal{F}(X,x;t), \tag{59}$$

where the parameters A, B, C, and D are

$$A := r_3 - r_2 + \frac{r_4 - r_1}{2}, \quad B := r_1 + r_4,$$

$$C := \frac{1}{2}[r_1 + r_4 + 2(r_2 + r_3)], \quad D := \frac{r_4 - r_1}{2}. \tag{60}$$

Using the change of variables

$$\hat{X} := X + \left(A - \frac{BD}{2C} \right) t - \frac{D}{2C} x \tag{61}$$

and

$$\mathcal{F}(X,x;t) = :\exp\left[-\frac{B}{2C}x - \frac{B^2}{4C}t \right] \hat{\mathcal{F}}(\hat{X},x;t), \tag{62}$$

one arrives at

$$\frac{\partial \hat{\mathcal{F}}(\hat{X},x;t)}{\partial t} = \left[C\partial_x^2 + \left(\frac{C}{4} - \frac{D^2}{4C} \right) \partial_{\hat{X}}^2 \right] \hat{\mathcal{F}}(\hat{X},x;t). \tag{63}$$

The boundary conditions for $\hat{\mathcal{F}}$ are

$$\hat{\mathcal{F}}(\hat{X},x=0;t) = \hat{\mathcal{F}}(\hat{X},x\to\infty;t) = 0. \tag{64}$$

The Green function $G(\hat{X},\hat{X}',x,x';t)$ for the equation (63) with the boundary conditions (64) is

$$G(\hat{X},\hat{X}',x,x';t) := \frac{1}{4\pi t\sqrt{CC'}} \exp[-(\hat{X}-\hat{X}')/(4C't)][\exp\{-(x-x')/(4Ct)\} - \exp\{-(x+x')/(4Ct)\}], \tag{65}$$

where $C' := (C/4) - D^2/(4C)$. Finally,

$$\mathcal{F}(X,x;t) = \exp\left(-\frac{B}{2C}x - \frac{B^2}{4C}t\right) \times \int_{x'=0}^{\infty}\int_{\hat{X}'=-\infty}^{\infty} dx'd\hat{X}' \times G(\hat{X},\hat{X}',x,x';t)\mathcal{F}(X',x';0)e^{(B/2C)x'}. \tag{66}$$

To obtain the solution for $\mathcal{E}$, one has to add a particular solution with the boundary conditions

$$\mathcal{E}(x=0)=1, \quad \mathcal{E}(x\to\infty)=0. \tag{67}$$

One can choose this particular solution to be translationally invariant (X independent). One is then led to

$$(B\partial_x + C\partial_x^2)\mathcal{E}^{\mathrm{P}}(x)=0, \tag{68}$$

the solution to which is

$$\mathcal{E}^{\mathrm{P}} = \exp(-Bx/C). \tag{69}$$

This particular solution is the same as that of Sec. II in the continuum limit.

ACKNOWLEDGMENT

M.A. would like to thank the research council of the University of Tehran, for partial financial support.

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